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ANALYTIC DISCS IN CONORMAL BUNDLES TO REAL SUBMANIFOLDS OF $\mathbb{C}^n$

BY

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ABSTRACT

The aim of this paper consists in finding some criterion for the existence of complex curves inside the conormal bundle to a CR manifold. Complex curves contained (or, more generally, attached) to conormal bundles to CR manifolds are propagators of microlocal regularity of CR functions (see [H-T] and [T]).

1. Introduction

It is proved in [B-F] that any complex curve γ in a pseudoconvex hypersurface $M \subset \mathbb{C}^n$ has a holomorphic *lift* γ^* in the conormal bundle $T^*_M\mathbb{C}^n$. It is very easy to adapt the argument of [B-F] to prove that in case M is a hypersurface no more pseudoconvex but having a constant number of negative (or positive) Levi eigenvalues, then any complex curve $\gamma \subset M$ running in the direction of the Levi kernel (that is verifying $T_z\gamma \subset \operatorname{Ker} L_M(p)$ for $p \in T^*_M\mathbb{C}^n$ and $z = \pi(p)$) has such a holomorphic lift γ^*. (If $p = (z,\zeta)$ is a point of $T^*_M\mathbb{C}^n$, and if X and $\bar{Y}$ are vector fields tangent to M holomorphic and antiholomorphic respectively, we define the *microlocal* Levi form by $L_M(p)(X,\bar{Y}) = \langle [X,\bar{Y}],\zeta\rangle$.)

We generalize here the above result to the case of a manifold $M \subset \mathbb{C}^n$ of higher codimension. This conclusion could not follow from the techniques of [B-F]. A partial result in this direction was also obtained in [Z] but under the stronger assumption of a constant Levi rank (whereas here only the number of negative, or positive, eigenvalues is assumed to be constant). Note also that since through

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the canonical projection we have an identification $T_p^{\mathbb{C}}T^*_M\mathbb{C}^n \xrightarrow{\pi'} \operatorname{Ker} L_M(p)$, then the hypothesis $T_z\gamma \subset \operatorname{Ker} L_M(p)$ is necessary for existence of a holomorphic lift γ^* through p.

Complex curves in M are crucial objects in many topics of CR geometry. Namely they are *propagators* of holomorphic extendibility according to the celebrated theorem by Hanges-Treves [H-T] and, conversely, their absence entails *flabbiness* of CR functions (or, more generally, cohomology classes of the tangential $\bar\partial$ system to M (cf. [B-F]). On the other hand, complex curves γ^* in $T^*_M\mathbb{C}^n$ are a more refined tool because they give a full description of the variation of directions of extendibility, to be explained in §3. To give a short hint of what is going on, let us point out that a system of lifts of maximal rank (that is equal to the codimension d of M) gives rise to a connection on $T^*_M\mathbb{C}^n$ over γ and, by duality, a connection on $T_M\mathbb{C}^n$. By means of the construction of discs of [B-Z1] this connection relates directions of CR extendibility over diffferent points of γ. The novelty of the conclusions in the present paper with respect to [B-Z1] (and, less closely, to [T]) is that propagation takes place along the full disc and not just along its boundary.

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2. Statements

We first begin with some notation and definitions. We denote by $z = (z', z'')$ the coordinates in $\mathbb{C}^n = \mathbb{C}^d \times \mathbb{C}^{n-d}$ and write $z = x + iy$. By M we shall denote a CR generic submanifold of $\mathbb{C}^n$ of codimension d and by $y' = h(x', z'')$ for $h = (h_j)_j$ (or extensively $y'_1 = h_1(x', z''), \dots, y'_d = h_d(x', z'')$) with $h(0) = \partial h(0) = 0$, a set of equations. We also write $r_j := -y_j + h_j$ and $r = (r_j)_j$. We give now a construction of a basis of vector fields of type $(1, 0)$ tangent to M. We set $A := -\partial_{z'} r$ that is, extensively,

$$A(x', z'') = \frac{-i}{2}\mathrm{id}_{d\times d} - \frac{1}{2}\partial_{x'} h(x', z'')$$

and put $B := (A^{-1}\partial_{z''} h)^t$; we also write $B = (b_{ij})_{ij}$. We then define a basis X_i by putting

$$X_i = \partial_{z_i} + \sum_{j=1}^{d} b_{ij}(x', z'')\partial_{z_j}, \quad i = d + 1, \dots, n. \tag{1}$$

Let $p = (z, \zeta_o) \in T^*_M\mathbb{C}^n$ with $\zeta_o = \partial r_o(z)$, where $r_o = 0$ is an equation of M, let $X = \sum a_i(z)\partial_{z_i}$ and $Y = \sum b_i(z)\partial_{z_i}$ be two holomorphic tangent vector fields. Then we denote by $L_M(p)(X, \bar{Y}) = \langle [X, \bar{Y}], \zeta_o\rangle$ the Levi form of M with respect to the conormal p. By the Cartan formula we have $L_M(p)(X, \bar{Y}) = \sum_{i,j} \partial_{z_i}\bar{\partial}_{z_j}(r_o)(z)a_i(z)\bar{b}_j(z)$. We first prove some preliminary results.

Lemma 1: *Let M be a CR manifold of equations $y' - h(x', z'') = 0$ with $h(0) = 0, \partial h(0) = 0$, and let γ be a complex curve contained in M whose tangent vector is in the kernel of the Levi form of M. Then*

B is holomorphic along γ.

In particular, if $\partial_{x'} h|_\gamma \equiv 0$ then $\partial_{z''}h|_\gamma$ is holomorphic.

Proof: We fix coordinates such that γ is the z_n axis. We have

$$\bar{\partial}_{z_n}(A^{-1}\partial_{z''}h) = A^{-1}(\bar{\partial}_{z_n}\partial_{x'}(h)A^{-1}\partial_{z''}(h) + \bar{\partial}_{z_n}\partial_{z''}(h)).$$

On the other hand, with the notation $A^{-1} = (c_{ij})_{ij}$, we have

$$(2) \qquad \sum_{\alpha\beta}(\bar{\partial}_{z_n}\partial_{x_\alpha}(h_j)c_{\alpha\beta}\partial_{z''_i}(h_\beta)) + \bar{\partial}_{z_n}\partial_{z''_i}(h_j) = L_M(\partial r_j)(\bar{X}_n, X_i),$$

where the vector fields X_n and X_i are those of (1). Since $X_n \in \operatorname{Ker} L_M(\partial r_j)|_\gamma$ for any j, we conclude that $L_M(\partial r_j)(\bar{X}_n, X_i) = 0$ for all j and i, and therefore $\bar{\partial}_{z_n}(A^{-1}\partial_{z''}(h))|_\gamma \equiv 0$. ∎

Lemma 2: *Let M be a CR manifold, and suppose further that the z_n axis lies in M and that in a neighborhood of 0 the equations of M are of the form $y' = h(x', z'')$ with $h(0) = 0$, $\partial h(0) = 0$ and with the matrix $\partial_{x'}h(0, z_n)$ real analytic with respect to z_n and $\bar{z}_n$. Then there exist new holomorphic coordinates in which we have*

$$(3) \qquad \frac{\partial^{\alpha+1}h}{\partial_{x'}\partial^\alpha_{z_n}}(0) = 0 \qquad \textit{for all } \alpha.$$

Remark: In [B-E-R] there are *normal coordinates* in which the analogous form of (3) (and more) holds for any z''_j instead of the single z_n under the stronger assumption of real analyticity of each h_j (in all arguments). In the present paper the result is specified for the distinguished direction z_n.

Proof: We use

$$\begin{cases} z' - \varphi(z', z_n) \\ z'' \end{cases}$$

with $\varphi(0,z_n)\equiv 0$ and $\partial\varphi(0,0)=0$, as new coordinates. The equations of M take the form

$$(4)\qquad \frac{z'+\varphi(z',z_n)-\bar z'-\overline{\varphi(z',z_n)}}{2i}=h\Big(\frac{z'+\varphi(z',z_n)+\bar z'+\overline{\varphi(z',z_n)}}{2},z''\Big).$$

We differentiate (4) with respect to x' and get for $z_j=0\ \forall j<n$

$$(5)\quad \frac{\partial_{x'}\varphi(0,z_n)-\partial_{x'}\bar\varphi(0,\bar z_n)}{2i}=\partial_{x'}h(0,z_n,\bar z_n)\Big(1+\frac{\partial_{x'}\varphi(0,z_n)+\partial_{x'}\bar\varphi(0,\bar z_n)}{2}\Big).$$

We solve (4) with respect to $z'-\bar z'$ and recall our eventual goal (3). Because of the assumption on real analyticity, this latter is equivalent to the fact that (5), when evaluated for $\bar z_n=0$, is identically 0, that is

$$\frac{\partial_{x'}\varphi(0,z_n)}{2i}-\frac{\partial_{x'}\bar\varphi(0,0)}{2i}=\partial_{x'}h(0,z_n,0)\Big(1+\frac{\partial_{x'}\varphi(0,z_n)+\partial_{x'}\bar\varphi(0)}{2}\Big).$$

Assuming $\partial\varphi(0)=0$ we have that

$$\partial_{x'}\varphi(0,z_n)=(\frac{1}{2}(-i+\partial_{x'}h(0,z_n,0)))^{-1}\partial_{x'}h(0,z_n,0).$$

Thus we choose

$$\varphi(z',z_n)=(\frac{1}{2}(-i+\partial_{x'}h(0,z_n,0)))^{-1}\partial_{x'}h(0,z_n,0)z'$$

and the proof is complete. ∎

We prove our main result about existence of holomorphic lifts. As already pointed out in the introduction, the CR structure of $T^*_M\mathbb{C}^n$ is identified via π' not to the whole CR structure of M but only to the Levi kernel. Thus if M is Levi non-degenerate, there are no curves γ^* in $T^*_M\mathbb{C}^n$ though they can well exist in M (as is the case of $\gamma=\{\tau(1,1,0)\tau\in\Delta\}$ in the hypersurface of $\mathbb{C}^3$ defined by $y_3=|z_1|^2-|z_2|^2$). On the other hand, by the uniqueness of the (small) lifts (cf., e.g., [B-R-T]) the space of lifts, which inherit from $T^*_M\mathbb{C}^n$ a linear structure, has at most dimension d. The conclusion of the next statement is therefore most satisfactory.

THEOREM 3: *Let M be a generic C^2 submanifold of $\mathbb{C}^n$ of codimension d and suppose that M has a constant number of negative (or positive) Levi eigenvalues for each point of T^*_MX in a neighborhood of a fixed p in $T^*_M\mathbb{C}^n\setminus\{0\}$. Let γ be a complex curve contained in M whose tangent space belongs to the Levi kernel of M with respect to all conormals. Then the space of germs of holomorphic lifts γ^* of γ in $T^*_M\mathbb{C}^n$ has dimension d.*

Proof: As in the proof of Lemma 1 we assume that γ coincides with the z_n axis and we restrict our attention to a piece of γ, say $|z_n|\le 1$. We fix in $T^*_M\mathbb{C}^n$ a point, say $p = (0; \partial r_1(0))$ for $r_1 = -y_1 + h_1$. We recall our central hypothesis that the number of negative (and hence of semipositive) eigenvalues is constant in a neighborhood of p. In particular, we can split $T^{(1,0)}M$ in two bundles $T^{(1,0)}M = \mathcal{S}^{<0} \oplus \mathcal{S}^{\geq 0}$, orthogonal to each other with respect to $L_M(\partial r_1)$ such that $L_M(\partial r_1)|_{\mathcal{S}^{<0}} < 0, L_M(\partial r_1)|_{\mathcal{S}^{\geq 0}} \geq 0$ and with $\mathcal{S}^{\geq 0} \supset \operatorname{Ker} L_M(\partial r_1)$. Let X be a section of $\mathcal{S}^{\geq 0}$ which extends ∂_{z_n} from γ to the whole M; we also write $X = \zeta_1 X_1 + \cdots + \zeta_{n-d} X_{n-d}$, where the X_i's are the basis of $T^{(1,0)}M$ introduced in (1). Differentiating with respect to x' we have therefore

$$\begin{aligned}\partial_{x'}\left(L_M(\partial r_1)(X, \bar{X})\right) = &\sum_{ij} \partial_{x'}(\zeta_i)\bar{\zeta}_j L_M(\partial r_1)(X_i, \bar{X}_j)\\ &+ \sum_{ij} \zeta_i \partial_{x'}(\bar{\zeta}_j) L_M(\partial r_1)(X_i, \bar{X}_j)\\ &+ \sum_{ij} \zeta_i \bar{\zeta}_j \partial_{x'}(L_M(\partial r_1)(X_i, \bar{X}_j)).\end{aligned} \tag{6}$$

When we evaluate along γ, then we get 0 in (6). In fact, the first two terms on the right hand side are 0 because $X|_\gamma \in \operatorname{Ker} L_M(\partial r_1)$, whereas the third is 0 because of the constancy of negative eigenvalues. We have in conclusion

$$\partial_{x'}(L_M(\partial r_1)(X_n, \bar{X}_n))|_\gamma = 0.$$

Repeating this argument for all conormals $\partial r_1 + \sum_{j=2,\ldots,d} \epsilon_j \partial r_j$ (ϵ_j small), and then taking linear combinations, we conclude

$$\partial_{x'}(L_M(\partial r_j)(X_n, \bar{X}_n))|_\gamma = 0 \quad \forall j. \tag{7}$$

Recall our notation $A(x', z'') = -(i + \partial_{x'}h(x', z''))/2 = (c_{\alpha\beta})_{\alpha\beta}$; we have by (2)

$$(L_M(\partial r_j)(X_n, \bar{X}_n))_{j=1,\ldots,d} = \partial_{z_n}\bar{\partial}_{z_n} h_j(x', z'') + \Re e \sum_{\alpha\beta} \bar{\partial}_{z_n}\partial_{x_\alpha}(h_j) c_{\alpha\beta} \partial_{z_n}(h_\beta).$$

Thus by differentiation in x', and by evaluating along γ, (7) becomes

$$\partial_{z_n}\bar{\partial}_{z_n}\partial_{x'} h_j + \Re e \bar{\partial}_{z_n}\partial_{x_\alpha}(h_j) c_{\alpha\beta} \partial_{z_n}\partial_{x'}(h_\beta) = 0. \tag{8}$$

Hence $\partial_{x'}h$ satisfies an elliptic (non-linear) system. It follows that $\partial_{x'}h$ must be a real analytic function in $z_n, \bar{z}_n$. By a complex coordinate change as in Lemma 2 we can assume that $\partial^\alpha_{z_n}\partial_{x'}h(0) = 0$ for any $\alpha > 0$. Write $\partial_{x'}h(0, z_n) = \sum_{\alpha\beta}^\infty d_{\alpha,\beta} z_n^\alpha \bar{z}_n^\beta$. We can simply prove by induction that $d_{\alpha,\beta} = 0$ for all α, β. This is true for $\alpha + \beta = 1$ and we suppose it true for $\alpha + \beta < k$; let us prove that it also holds for $\alpha+\beta=k$. In fact, for $\alpha+\beta=k$ we get by (8), by Lemma 2 and by the inductive hypothesis

$$\alpha\beta d_{\alpha\beta}=i\sum_{l,m\geq 0}(l(\beta-m)d_{l,m}d_{\alpha-l,\beta-m}+(\alpha-l)md_{l,m}d_{\alpha-l,\beta-m})=0.$$

We thus conclude that $\partial_{x'}h\circ\gamma\equiv 0$; in particular, $T^{(1,0)}M=\mathrm{Span}\{\partial_{z''}\}$. Recall that we are assuming that $\partial_{z''}h|_\gamma$ is holomorphic by Lemma 1. Thus in conclusion by letting

$$\gamma_j^*:=\partial(-y_j+h_j)|_\gamma,$$

we get a system of d independent holomorphic lifts. ∎

Remark: In the construction of the lifts γ_j^*, we need to shrink γ in many steps in order to get *normalized equations* and also to enjoy the hypothesis that the number of negative eigenvalues is constant. In case this latter assumption of constancy holds in a neighborhood of a global section of $T_M^*\mathbb{C}^n$ all over γ, we may patch together lifts over small pieces of γ to get a global lift. For this we use the crucial and elementary fact (see, e.g., [B-Z]) that the (small) lift through a prescribed point of $T_M^*\mathbb{C}^n$ is unique if it exists.

3. Application to propagation

Let M be a generic manifold in $\mathbb{C}^n$ and let CR_M be the CR functions on M, that is the continuous solutions f of the system $\bar{X}_jf=0$ where $\bar{X}_j$ is a basis of $(0,1)$ tangential vector fields to M. We recall from [B-T] that if we fix $z_o\in M$, then there is a controlled neighborhood of z_o in M, that we still denote by M, such that any CR function can be uniformly approximated on M by polynomials. We suppose that M is of class $C^{k,\alpha}$ and assume that there is a complex curve $\gamma\subset M$ which verifies $T\gamma\subset\bigcap_{j=1,\dots,d}\mathrm{Ker}\,L_M(\partial r_j)|_\gamma$, where $r_j=0$, $j=1,\dots,d$ is a system of independent equations for M. We assume that the number of negative (or positive) Levi eigenvalues of M is constant in a neighborhood of a section of $T_M^*\mathbb{C}^n$ over γ. We recall from §2 that in this situation there is a system of (global) holomorphic lifts $\{\gamma_j^*\}_{j=1,\dots,d}$ of rank d (cf. Remark 4). Vectors in the space of forms such as the γ_j^* will be written as row vectors $\gamma_j^*=(\gamma_{ji}^*)_{i=1,\dots,d}$. With the aid of the above basis of $T_M^*\mathbb{C}^n$, we can identify the dual bundle $T_M\mathbb{C}^n$ (that is the normal bundle to M) to $M\times\mathbb{R}^d$ by setting

$$[v]\mapsto(\Re e\langle\gamma_j^*,v\rangle)_j,$$

where $[v]$ denotes the equivalence class modulo TM. We also put

$$\Gamma^* = (\gamma^*_{ji})_{ji}.$$

We define a connection on $T^*_M\mathbb{C}^n$ above γ as follows. Along with z_o, we fix another point $z_1 \in \gamma$; for the sake of simplicity, we also suppose $\Gamma^*(z_1) = \mathrm{id}_{d\times d}\times 0_{d\times(n-d)}$. Then in this situation we define a morphism Φ as the one which makes the following diagram commutative,

$$\begin{array}{ccc} (T_M\mathbb{C}^n)_{z_1} & \xrightarrow[\Phi]{\sim} & (T_M\mathbb{C}^n)_{z_o} \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_o} \\ \mathbb{R}^d & \xrightarrow[\sim]{\Gamma^*(z_1)} & \mathbb{R}^d \end{array}$$

We will say that a germ of manifold M_o or M_1 is an extension of M which points to the normal unit direction v_o or v_1 at z_o or z_1, if M_o (M_1) is a manifold with boundary M such that $(T_M M_o)_{z_o} = \mathbb{R}^+ v_o$ or $(T_M M_1)_{z_1} = \mathbb{R}^+ v_1$.

THEOREM 4: *In the above situation, let f be a CR function on M which extends at z_1 to a manifold M_1 which points to v_1. Then for any ϵ there is a manifold M_o which points to a direction v_o which verifies $|v_o - \Gamma^*(z_1)v_1| < \epsilon$, such that any CR function on $M \cup M_1$ extends as a CR function to $M \cup M_o$.*

Proof: We just need to repeat step by step the proof of [B-Z1, Th. 3]. Note that in the aforementioned theorem only pairs of points z_o and z_1 of $\partial\gamma$ could be treated. ∎

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